Onset of Thermosolutal Convection in a Liquid Layer Having Deformable Free Surface – II. Overstability

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The onset of thermosolutal convection driven by a temperature and concentration dependent surface tension is investigated for a thin layer of fluid having a deformable free surface. It is shown that there exist two Crispation numbers (Cr) for oscillatory modes of instability. It is further shown that Cr and the frequency of the oscillatory mode are strongly coupled for large values of Cr. It is found that Cr destabilizes the system for both cases, salted from below and salted from above.

1. Introduction

The effect of a free surface deformation on the onset of a surface tension driven instability in a horizontal thin liquid layer subject to a vertical temperature and concentration gradient is examined using linear stability theory. In [1] we had studied the onset of thermosolutal convection in the frame of stationary convection and obtained the existence of two Crispation numbers Cr_1 and Cr_2. The other possibility of instability, setting in as overstability, was not explored in that study.

The aim of the present paper is to study the effect of the free surface deformation on the initiation of oscillatory instability.

2. The Problem and its Solution with a Numerical Procedure

The governing equations of W, θ, and ϕ for linear stability are given by (cf. [1], (13a–c))

$$(D^2-a^2)(D^2-a^2-\omega P_r^{-1})W=0,$$
$$(D^2-a^2-\omega)\theta=-W,$$
$$[\tau(D^2-a^2)-\omega]\phi=W. \qquad (1\,\mathrm{a–c})$$

The associated boundary conditions are given by (cf. [1], (14a–d) and (15a–e))

$$W=DW=\theta=\phi=0 \quad \text{at} \quad z=0 \qquad (2\,\mathrm{a–d})$$

$$W-\omega Z=0, \qquad (3\,\mathrm{a–e})$$
$$\mathrm{Cr}(D^2-3a^2-\omega P_r^{-1})DW-a^2(B_0+a^2)Z=0,$$
$$(D^2+a^2)W+a^2(M\theta-M'\phi)-a^2(M+M')Z=0,$$
$$(D+B_i)\theta-B_iZ=0,$$
$$(D+B_i')\phi+B_i'Z=0, \qquad \text{at} \quad z=1.$$

Since in this study we want to explore the effect of surface deformation on oscillatory instability, we shall solve (1)–(3) for $\omega\neq0$. For the boundary conditions (2a, b) at $z=0$ and (3a, b) at $z=1$ we easily find from (1a)

$$W=A_1[S_{az}-K_0C_{az}-(a/b)S_{bz}+K_0C_{bz}], \qquad (4)$$

where

$$S_i=\sinh i, \quad C_i=\cosh i, \quad (i=a, b, d, e, \text{etc.}),$$
$$b=[a^2+\omega/P_r]^{1/2},$$
$$K_0=\frac{(\omega\mathrm{Cr})[b_1C_a-2a^2C_b]+a(B_0+a^2)[S_a-(a/b)S_b]}{(\omega\mathrm{Cr})[b_1S_a-2abS_b]+a(B_0+a^2)[C_a-C_b]},$$
$$b_1=2a^2+(\omega/P_r)$$

and A_1 is an arbitrary constant.

Similarly we can obtain the solution for θ and φ by solving the sets (1b, 2c, 3d) and (1c, 2d, 3e) as

$$\theta=(A_1/\omega)[K_1S_{dz}+(K_0/(1-P_r))C_{dz}+S_{az}-K_0C_{az}+(a/b)(P_r/(1-P_r))S_{bz}-(K_0P_r/(1-P_r))C_{bz}], \qquad (5)$$

$$\varphi=(A_1/\omega)[(K_2/\tau)S_{ez}-(K_0\tau/(\tau-P_r))C_{ez}-\{S_{az}-K_0C_{az}+(aP_r/(\tau-P_r)b)S_{bz}-(K_0P_r/(\tau-P_r))C_{bz}\}], \qquad (6)$$

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where $d=(a^2+\omega)^{1/2}$, $e=(a^2+(\omega/\tau))^{1/2}$,

$$K_1=[(K_0 d/(P_r-1)) S_d - a C_a + K_0 a S_a - (a P_r/(1-P_r)) C_b + (K_0 b P_r/(1-P_r)) S_b + B_i\{(K_0/(P_r-1)) C_d - (a/b(1-P_r)) S_b + (K_0/(1-P_r)) C_b\}]/[d C_d + B_i S_d],$$

$$K_2=[(K_0 e \tau^2/(\tau-P_r)) S_e + \tau\{a C_a - K_0 a S_a + (a P_r/(\tau-P_r)) C_b - (K_0 b P_r/(\tau-P_r)) S_b\} + B_i'\{(K_0\tau^2/(\tau-P_r)) C_e + (a\tau^2/b(\tau-P_r)) S_b - (K_0\tau^2/(\tau-P_r)) C_b\}]/[e C_e + B_i' S_e].$$

So far we have not used the boundary condition (3c) in obtaining the solutions for W, θ, and φ. Now use of (4–6) in (3c) gives an eigenvalue relationship

$$f(a, \omega, \tau, M, M', B_0, B_i, B_i', P_r)=0 \tag{7}$$

whose explicit form is given by

$$A+BM-CM'=0, \tag{8}$$

where

$$A=a[2a^2(S_a-K_0 C_a)+b_1\{-(a/b) S_b+K_0 C_b\}],$$

$$B=a^2[K_1 S_d+(K_0/(1-P_r)) C_d+(a P_r/b(1-P_r)) S_b - (K_0/(1-P_r)) C_b+(a/b) S_b],$$

and

$$C=a^2[(K_2/\tau) S_e-(K_0\tau/(\tau-P_r)) C_e - (a\tau/b(\tau-P_r)) S_b+(K_0\tau/(\tau-P_r)) C_b].$$

For a non-stationary neutral state we have $\omega=i\omega_1$, where ω_1 is real. Thus, A, B, and C are in general complex and can be expressed in the form a_3+ia_4, a_1+ia_2 and a_5+ia_6, respectively. The explicit form of $a_1, a_2, \ldots, a_6$ can be obtained after a laborious but straight forward algebric calculation, and as these expressions are quite lengthy, so we avoid them here. They can be obtained on request from the authors. Without going into details one can obtain from (8)

$$a_3+ia_4+M(a_1+ia_2)-M'(a_5+ia_6)=0. \tag{9}$$

This equation gives

$$M=\frac{M'(a_1a_5+a_2a_6)-a_3a_1-a_4a_2+i[M'(a_1a_6-a_2a_5)-a_4a_1+a_2a_3]}{a_1^2+a_2^2} \tag{10}$$

Since M is real, we have

$$M'(a_1a_6-a_2a_5)-a_4a_1+a_2a_3=0. \tag{11}$$

Thus

$$M=\frac{M'(a_1a_5+a_2a_6)-a_3a_1-a_4a_2}{a_1^2+a_2^2}. \tag{12}$$

Assigning arbitrary values to the parameters M', B_0, B_i, B_i', τ, P_r, Cr, ω_1, and a one can calculate M from (12) satisfying (11), and the solution is acceptable when (11) is satisfied to within a tolerance error of 10^{-5}. To find the minimum value of $M=M_c$ we have fixed all the parameters except ω_1 and a. At first, a is allowed to vary and we note the minimum value of M, say $M=M_{1c}$, for a particular value of ω_1. Then we supply the next value to ω_1 and vary a to obtain M_{2c} and compare M_{1c} with M_{2c}. In this way finally we obtain M_c, which is the smallest among all M_{ic} $(i=1,2,\ldots)$ for critical values of $\omega_1(=\omega_c)$ and $a(=a_c)$. It should be pointed out here that for large values of the wave number a our numerical tolerance is not fulfilled, so we have discarded those results and conclude that the neutral state is stationary for those values of a.

3. Results and Discussion

Figure 1 shows the variation of M with a for several values of Cr when the other parameters are fixed. It should be pointed out here that M will be unchanged for all values of Cr $\leqq 0.00002$. We designate this Cr as Cr_1 (see [1]). It is clear from Fig. 1 that, as Cr increases past Cr_1, a local minimum is formed for small values of a but M will be smallest for large values of a. For $\mathrm{Cr}=\mathrm{Cr}_2$ a minimum value for M can be obtained at two different values of a. On further increase of Cr past Cr_2, M will be smallest for small values of a. It is evident from Fig. 1 that no oscillatory instability will exist for $a\to 0$. This results confirms our findings in [3]. Figures 2 and 3 depict the change of M_{ic} with ω_1. It is clear that the positive or negative value of M_{ic} depends on the selection of the frequency range of ω_1, although for the entire range of ω_1 the smallest value of M_{ic} (i.e. M_c) will always be negative. One may interpret this negative value of M_c as heating from above [4]. Now we like to emphasize that by selecting

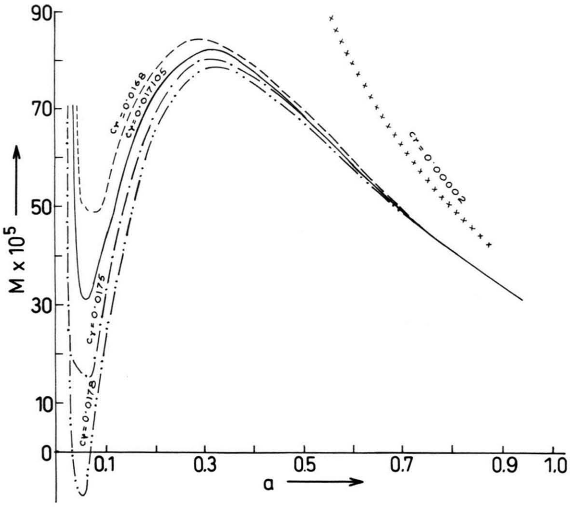

Fig. 1. Variation of M with a for different values of Cr when $P_r = 0.001$, $\tau = 0.07$, $B_0 = 0.1$, $B_i = B_i' = 0$, $M' = 4$ and $\omega = 0.001$.

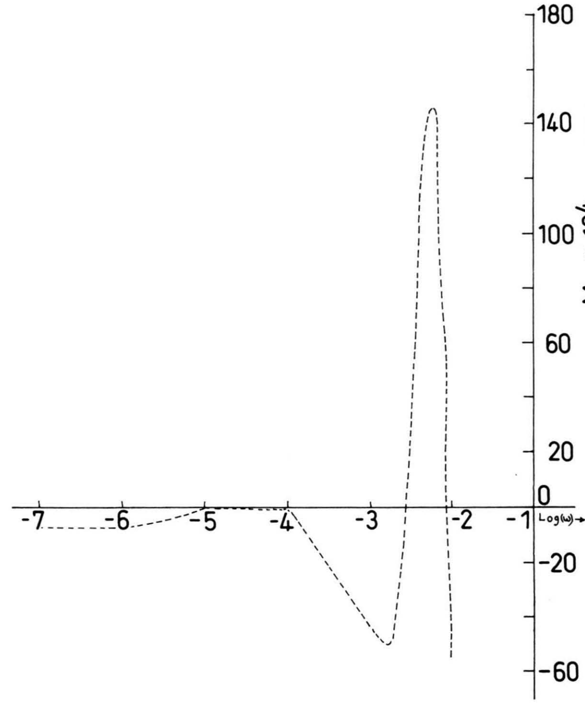

Fig. 3. Variation of M_{ic} with log (ω) for $P_r = 0.001$, $\tau = 0.01$, $B_0 = 0.1$, $B_i = B_i' = 0$, Cr = 0.001 when $M' = 20$.

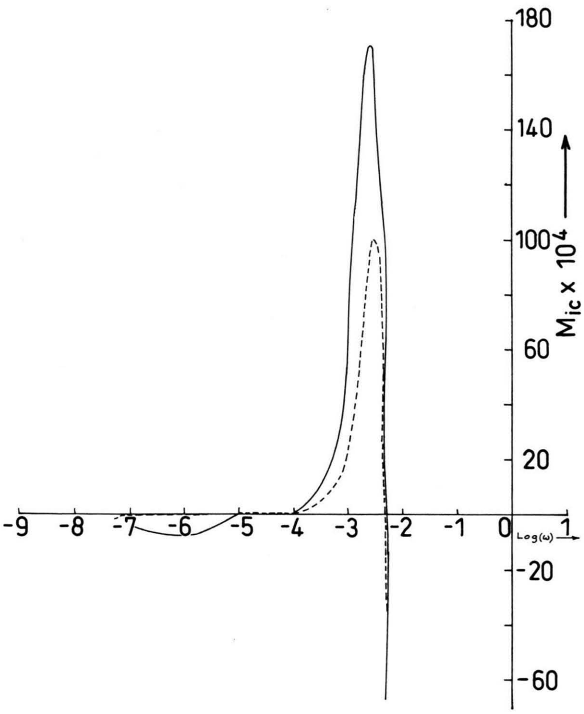

Fig. 2. Variation of M_{ic} with log (ω) for different M' when $P_r = 0.001$, $\tau = 0.01$, $B_0 = 0.1$, $B_i = B_i' = 0$, Cr = 0.001. — for $M' = -25$ and - - - for $M' = -5$.

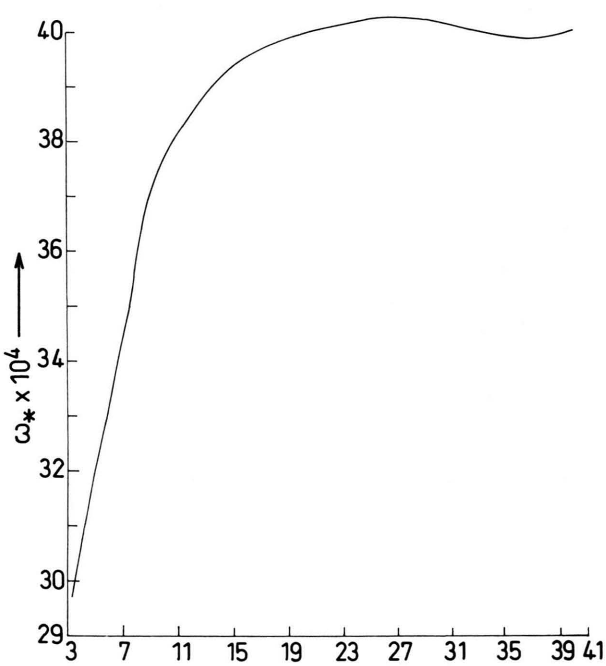

Fig. 4. Variation of ω with Cr when $\tau = 0.01$, $P_r = 0.001$, $B_0 = 0.1$, $M' = 20$ and $B_i = B_i' = 0$.

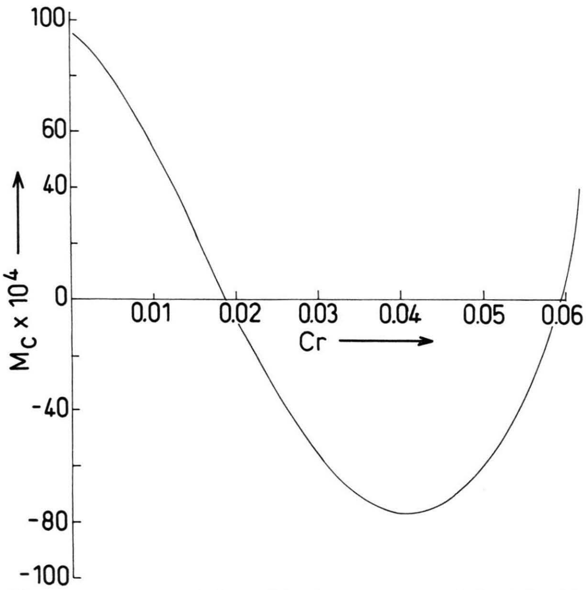

Fig. 5. Variation of M_c with Cr when $\tau = 0.01$, $M' = 20$, $P_r = 0.001$, $B_0 = 0.1$, $B_i = B_i' = 0$ and $\omega = \omega_* = 0.003992$.

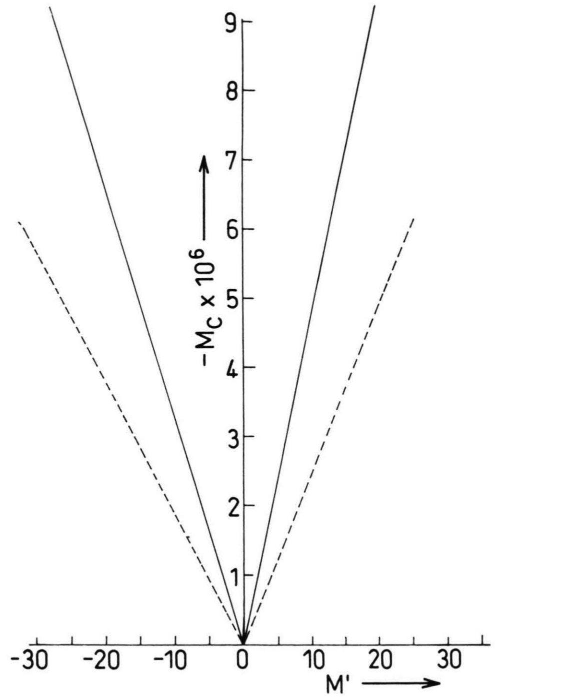

Fig. 6. Variation of $-M_c$ with M' when $P_r = 0.001$, $\tau = 0.01$, $B_0 = 0.1$, $B_i = B_i' = 0$. — for Cr = 0.001 and - - - for Cr = 0.01.

a particular frequency range for ω_1 it is possible to suppress the onset of the oscillatory mode of instability either for the system heated below or from above. Figure 2 shows the changes of M_{ic} with ω_1 for salted from below, whereas Fig. 3 shows the same for salted from above. Figure 4 represents the variation of ω_* (the value of ω_1 for which $|M_c|$ is minimum) with Cr for a particular set of values of all other parameters. It is clear from the graph that ω increases with Cr up to a certain value of Cr (say Cr*), beyond which ω_* oscillates within a certain range. The physical explanation for the oscillation of ω_* for Cr > Cr* is not clear to us. However, Fig. 5 depicts the variation of M_c with Cr for fixed values of M', P_r, B_0, τ, B_i, and B_i'. Here we have taken that value of ω_* for which Cr = Cr* as predicted in Figure 4. Figure 6 shows the change of $-M_c$ with M' for different Cr keeping all other parameters fixed. It is clear that the increase of Cr destabilizes the system for both cases, salted from below or above.

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